Calculation of Laminar Separation Bubbles in the Wake Inflation/Deflation Regime

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A new interacting boundary-layer method is developed for calculating two-dimensional incompressible laminar separation bubbles. The method relies on a completely consistent mixed boundary value treatment of the outer inviscid flow. For attached flows, the new method is found to converge as fast as the classical Cauchy-Hilbert method and to yield the same solutions. For separated flows, the new method is found to converge two to three times faster than a comparable Cauchy-Hilbert method. The physical asymptotic theories that underlie this new numerical method will be discussed, in so far as they motivate the method.

Nomenclature

f	= boundary-layer scaled airfoil shape
F, V	= velocity components in Görtler variables
Im	= imaginary part
p	= pressure
Re	= Reynolds number
u, v	= inviscid flow streamwise and normal velocity components
Ue	= inviscid edge velocity
x_0	= separation point in triple-deck theory/point of boundary condition switch in interacting boundary-layer model
α	= triple-deck constant
β	= pressure gradient parameter
δ	= boundary-layer displacement thickness
λ	= boundary-layer shear stress
Ω	= airfoil thickness (half-thickness-to-chord ratio)
η, ξ	= Görtler variables

Superscript

$(\ )'$	= perturbed quantities

I. Introduction

At high Reynolds numbers, the flow near a laminar separation point is often governed by the triple-deck theory of Stewartson and Williams,[1] Messiter,[2] and Neiland.[3] For incompressible flows past bluff bodies, this theory has been adapted to the Kirchhoff[4] model of inviscid separation by Sychev.[5] In this theory, the triple-deck fixes the viscous separation point a small distance downstream of the inviscid smooth separation point of the Kirchhoff model. Smith[6] has confirmed this local theory through numerical calculations. The primary result of Smith's[6] work is that the triple deck fixes the magnitude of a $Re^{-1/16}$ square root singularity in the inviscid flow pressure gradient just upstream of the viscous separation point. Since the pressure gradient singularity entering the triple deck must be small, the inviscid separation point must be at a point where the pressure gradient at separation is zero to leading order, i.e., a smooth separation point. As the airfoil thickness decreases, this state of affairs persists until the airfoil thickness reaches a scale of $Re^{-1/16}$. On the $Re^{-1/16}$ thickness scale, Cheng and Smith[7] have shown that the separation point departs from the smooth separation point. The location of the separation point is determined primarily by the flow within the triple deck. In this asymptotic theory, the issue of reattachment still needs to be addressed, and the reader is referred to Smith[8,9] and Cheng and Smith[7] for further discussion on this issue.

The primary result of the theories of Sychev,[5] Smith,[6] and Cheng and Smith[7] is that, at high Reynolds numbers, the base inviscid solution is nonunique and is fixed by a local viscous condition at the separation point. In a sense, this viscous triple-deck criterion at the separation point plays a role that is analogous to the role of the viscous Kutta condition in lifting inviscid flows but in the context of an inviscid separated flow. The reattachment point also may exert considerable influence, but this is not understood at present. When the airfoil is relatively thin, of thickness $\mathcal{O}(Re^{-1/16})$, small changes in the airfoil geometry may result in large changes in the separation bubble size and shape. In fact, near a certain critical airfoil thickness, the separation bubble may undergo an inflation/deflation with a slight change in airfoil thickness. The airfoil inflation is, quite literally, a rapid steady-state inflation of the separation bubble from a body-scale separation bubble to a very large-scale separation bubble. The deflation is the reverse of this process. Cheng and Smith[7] also have observed nonunique solutions in the $Re^{-1/16}$ regime that naturally lead to stall hysteresis; this point will not be discussed in the present study.

The issue addressed in this study is how to calculate incompressible laminar separation using the interacting boundary-layer model. The above discussion suggests that careful attention must be paid to the flow near the separation point, since the local viscous separation ultimately controls the entire downstream eddy structure. Rothmayer and Davis[10] developed an efficient interacting boundary-layer model for laminar separation by using a formulation for the outer inviscid flow that possessed a square root singularity just upstream of the separation point. This square root singularity triggered, and encouraged, the triple-deck free interaction at separation but was eventually smoothed out by the displacement interaction effect. One major drawback of the method of Rothmayer and Davis[10] was that an unrealistic Kirchhoff semi-infinite eddy model was used downstream of the separation point. The present study corrects this deficiency by replacing the Kirchhoff eddy model with a full numerical solution within the eddy. This effect is incorporated into the method of Rothmayer and Davis.[10] The new interacting boundary-layer model developed in this study is found to converge at roughly the same rate as the classical Cauchy-Hilbert model for attached flows. In addition, the results of the Cauchy-Hilbert method and the present model are identical. For separated flows, the present model converges two to three times faster than the

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classical Cauchy-Hilbert method, again with identical results. It should be noted that this new method can be extended easily to flows past bluff bodies and past asymmetric lifting airfoils via the newly developed method of Rothmayer[11] for asymmetric bluff-body interacting boundary layers. The present method also is expected to have considerable relevance to the computation of turbulent separation bubbles, since these bubbles are often short scaled and are not at constant pressure.

II. Basic Interacting Boundary-Layer Model

The fundamental problem considered in this study is the two-dimensional incompressible flow past a thin airfoil aligned with an oncoming uniform stream. The physical problem addressed here is shown in Fig. 1.

As discussed in Sec. I, the initial stages of trailing-edge laminar separation at high Reynolds numbers occur on asymptotically thin airfoils, in the sense that the airfoil thickness generating a given bubble tends to zero as some power of the Reynolds number as the Reynolds number tends to infinity. In addition, the portion of the triple-deck within the boundary-layer is contained within the boundary-layer equations. If the theory of Cheng and Smith[7] is correct, then the equations governing the eddy and shear layer combination also are a subset of the boundary-layer equations. This result also applies for the thin Sadovskii[12] eddies of Smith's[9] model. Therefore, it seems reasonable, in the wake inflation/deflation regime, to use boundary-layer equations within the boundary layer, shear layer, and predominantly inviscid eddy. This would certainly be valid for triple-deck separation bubbles, and it is a likely candidate for the more extensive eddies. This description might become invalid if transverse pressure gradients become important in the eddy, as in a Prandtl-Batchelor type of model or near reattachment. However, should transverse pressure gradients turn out to be important, they can be included easily in the present model at a later date. For now, the separation bubble, boundary layer, and shear layer are assumed to lie in a region that can be described with boundary-layer equations. The boundary-layer region is allowed to interact with the outer inviscid flow via the displacement interaction effect. In the boundary layer, the governing equations are nondimensionalized with the freestream velocity and the chord length of the biconvex airfoil. The variables then are scaled with boundary-layer scaling laws and cast first in Prandtl transposed variables and then in Görtler variables, yielding

$$(\xi,\eta) = \left[\frac{x^*}{L}, \frac{Re^{1/2}(y^*/L) - f(x)}{\sqrt{2\xi}}\right] \tag{1}$$

where $f(x)$ is a boundary-layer scaled airfoil thickness:

$$f(x) = Re^{1/2}\frac{f^*(x^*)}{L} \tag{2}$$

Following the derivation of Davis and Werle,[13] the continuity equation is used to eliminate any ξ derivatives from the ξ momentum equation. The final boundary-layer model in standard Gortler variables is

$$V_\eta + F + 2\xi F_\xi = 0 \tag{3}$$

$$F_{\eta\eta} - VF_\eta + FV_\eta + F^2 + \beta = 0 \tag{4}$$

$$\beta_\eta = 0 \tag{5}$$

where β is the unknown pressure gradient parameter. Casting the equations in this form allows for the inclusion of transverse pressure gradients should they become important. The boundary conditions on this system are

$$F(\xi,0) = V(\xi,0) = 0 \qquad \text{over the body} \tag{6}$$

$$F_\eta(\xi,0) = V(\xi,0) = 0 \qquad \text{in the wake} \tag{7}$$

$$F \to Ue \qquad \text{as} \qquad \eta \to \infty \tag{8}$$

$$\beta \to 2\xi Ue Ue_\xi \qquad \text{as} \qquad \eta \to \infty \tag{9}$$

In addition, the following auxiliary V-matching condition will be needed:

$$V - \eta V_\eta \to \sqrt{2\xi}\,(Ue\delta)_\xi \qquad \text{as} \qquad \eta \to \infty \tag{10}$$

This set of boundary-layer equations is allowed to interact with the outer inviscid flow via the displacement interaction effect. The classical method for treating this interaction and the numerical method of Veldman[14] and Davis and Werle[13] will be discussed in the next section.

III. Quasisimultaneous Interacting Boundary-Layer Numerical Method

In the wake inflation/deflation regime, the airfoil and eddy are expected to be thin despite the presence of extremely long separation bubbles. The classical interacting boundary-layer model, used by Davis and Werle,[13] Veldman,[14] and many others, was initially used to model small trailing-edge, triple-deck separation bubbles. However, this model also should be valid in the wake inflation/deflation regime. The classical Cauchy-Hilbert interacting boundary-layer model allows the boundary layer to interact with the outer inviscid flow and describes the outer inviscid flow by the following thin airfoil equation:

$$Ue = 1 + \frac{Re^{-1/2}}{\pi}\int_0^\infty \frac{(\mathrm{d}/\mathrm{d}t)(Ue\delta + f)}{x - t}\,\mathrm{d}t \tag{11}$$

In this equation, the boundary-layer displacement interaction effect is felt by the outer inviscid flow as an effective injection velocity everywhere along the boundary layer and wake. The injection boundary condition used to derive the above equation is:

$$v(x,0) = Re^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(Ue\delta + f), \qquad 0 < x < \infty \tag{12}$$

where $v(x,y)$ is the $\mathcal{O}(1)$ normal velocity in the outer inviscid flow. One of the most efficient methods for solving this system of equations has been developed by Veldman[14] and Davis and Werle.[13] In the Veldman-Davis method, the continuity equation (3) and the two momentum equations (4) and (5) are inverted simultaneously using a block tridiagonal algorithm. The boundary conditions for Eqs. (6), (7), and (9) are easily implemented in this algorithm (see Ref. 13 for details). The Veldman-Davis method deals primarily with the treatment of the streamwise velocity matching condition [Eq. (8)], as follows:

$$V - \eta V_\eta \to \sqrt{2\xi}\left[\bigcirc\!\!\!\!\!\!\;(Ue\delta)\right]_\xi \qquad \text{as} \qquad \eta \to \infty \tag{13}$$

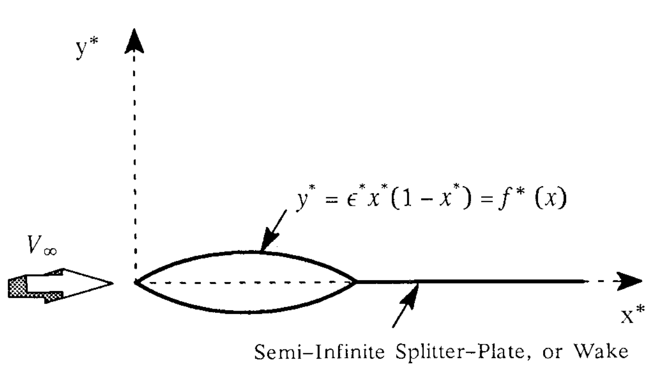

Fig. 1 Physical geometry.

Step 2:

$$\boxed{Ue} = 1 + \frac{Re^{-1/2}}{\pi} \int_0^\infty \frac{(\mathrm{d}/\mathrm{d}t)\left[\boxed{Ue\delta} + f\right]}{x-t} \, \mathrm{d}t \tag{14}$$

Step 1:

$$F \to \boxed{Ue} \quad \text{as} \quad \eta \to \infty \tag{15}$$

In the Veldman-Davis method, the inviscid surface speed and the boundary-layer displacement thickness are simultaneously eliminated from the outer-edge boundary conditions essentially in the manner of the two-step procedure shown in Eqs. (13-15). In order for this method to work, there must be a direct relationship between the inviscid surface speed and the boundary-layer displacement thickness of the form

$$Ue_i = \bar{C}_i + \bar{D}_i(Ue\delta)_i \tag{16}$$

where i is the gridpoint associated with x in Eq. (14). This relation affords a direct connection between Ue and δ and allows step 2 to be performed in the Veldman-Davis method [see Eqs. (13-15)]. In the Cauchy-Hilbert interacting boundary-layer model, the finite-difference form of the thin-airfoil equation, Eq. (14), can be manipulated into the form of Eq. (16) (see Ref. 13 for details). Most interacting boundary-layer studies are variations on these basic equations, although many use a different coupling algorithm for the pressure-displacement interaction.

IV. Massive Separation

Historically, interacting boundary-layer methods have been able to handle small-scale separation quite efficiently, even at large Reynolds numbers. However, as the extent of laminar separation increases, all methods, including interacting boundary-layer methods, tend to experience a drastic decrease in efficiency. An examination of the cause of the difficulties encountered in computing laminar separation requires an understanding of the physics of laminar separation. The high Reynolds number asymptotic theory for two-dimensional laminar separation is sufficiently advanced to provide such insights into the computational problem. As discussed in Sec. I, the basic theory of incompressible laminar separation was provided by Sychev[5] and Smith.[6] The Sychev-Smith theory essentially states that the basic inviscid solution past a body is nonunique and that a triple deck at the separation point selects among these nonunique inviscid solutions and chooses the one that is locally compatible with the triple-deck pressure gradients and the form of the separated shear layer emerging from the triple deck. In particular, the triple deck requires that the pressure approaching its upstream side be singular and have the form

$$p - p_e \sim -aRe^{-1/16}\lambda(x_0)^{9/8}(x_0 - x)^{1/2} + \ldots \tag{17}$$

where p is the surface pressure, p_e the local eddy pressure, x the arc length measured along the body surface, x_0 the separation point, and λ the boundary-layer shear stress; α is a constant fixed by the triple-deck structure at separation (see Ref. 6). Originally, Sychev[5] used the classical Kirchhoff[4] free-streamline model as an example of a separated inviscid flow that could support the triple-deck criterion of Eq. (17). Note that Eq. (17) states that any separation-point pressure gradient singularity present in the inviscid solution must be asymptotically small as the Reynolds number tends to infinity. The triple-deck criterion of Eq. (17) then fixes the viscous separation point a small distance downstream of the Kirchhoff smooth separation point. The Kirchhoff free-streamline theory used by Sychev[5] models the eddy as a constant-pressure deadwater region. Even in Smith's[9] model, where the body-scale eddy is no longer a deadwater region (or even a constant-vorticity region), the effect of the triple-deck criterion is still the same. The triple-deck criterion selects a compatible separated inviscid flow solution, effectively by setting the location of the separation point in the solution.

The crossover mechanism from the bluff-body eddies of Sychev[5] and Smith[6] to the smaller-scale, triple-deck trailing-edge separation bubbles is provided by the theory of Cheng and Smith.[7] They observed that the triple-deck criterion of Eq. (17) requires a fixed, but small, pressure gradient singularity at the separation point. As long as the airfoil thickness is larger than $Re^{-1/16}$, then the triple-deck criterion places the separation point at the smooth separation point. However, when the airfoil thickness decreases to $Re^{-1/16}$, the leading-order pressure on the airfoil is the same order of magnitude as the pressure perturbation required to produce the triple-deck criterion of Eq. (17). The general method of Cheng and Rott,[15] which is based on earlier work by Betz,[16] Carleman,[17] and Munk,[18] is used to generate the separated Kirchhoff solutions for flow past a thin airfoil. Upstream of separation, the pressure for a Kirchhoff free-streamline flow past an airfoil $y = Re^{-1/16}f(x)$ with the separation point at x_0 is found to be

$$p(x,0) \sim -Re^{-1/16}\frac{h}{\pi}\sqrt{x_0 - x}\int_0^{x_0}\frac{f'(t)\,\mathrm{d}t}{\sqrt{x_0 - t}(x-t)} \tag{18}$$

Downstream of the separation point, the eddy pressure is assumed to be at the freestream pressure for the open body-scale eddy (A in Fig. 2). The triple-deck criterion of Eq. (17) is applied to Eq. (18) to generate an equation for the location of the separation point (see Ref. 7). As the airfoil thickness is changed, the viscous criterion governing the location of the separation point x_0 eventually will produce a collapse, or deflation, of the long eddy A onto the shorter scale eddy B of Fig. 2. The inverse process will be referred to as wake inflation.

The above discussion indicates that laminar separation in the wake inflation/deflation regime and in the immediate vicinity of the separation point is triggered by a pressure gradient singularity just upstream of separation. The resulting triple-deck free interaction at the separation point then controls the entire downstream eddy structure by setting the level of the square root singularity in the inviscid flow at the separation point. Rothmayer and Davis[10] developed an interacting boundary-layer model for thin airfoil massive separation that took advantage of this concept. They reasoned that since the triple-deck free interaction was triggered by a square root singularity in an inviscid solution, using a numerical form of this square root singularity might trigger and promote the triple-deck free interaction in the interacting boundary-layer calculations. In their model, the outer inviscid flow was treated as a mixed boundary value problem with the following boundary conditions:

$$v'(x,0) = \frac{\mathrm{d}}{\mathrm{d}x}(Ue\delta + f), \qquad x < x_0 \tag{19}$$

$$u'(x,0) = 0, \qquad x > x_0 \tag{20}$$

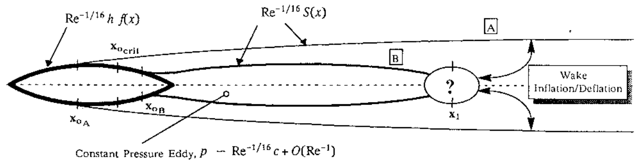

Fig. 2 Eddy geometry in the wake inflation/deflation regime: A is the open eddy calculated by Rothmayer and Davis,[10] B the closed eddy calculated in the present study. The separation points are indicated for both the long- and short-scale eddies. The critical separation point location for the inflation/deflation of the short-scale/long-scale eddy is denoted by $x_{0_{crit}}$.

where

$$(u,v) \sim (1,0) + Re^{-1/2}(u',v') + \dots \tag{21}$$

When the method of Cheng and Rott[15] is applied to the outer inviscid flow, the following equation is found for Ue:

$$Ue = 1 + \frac{Re^{-1/2}\sqrt{x_0 - x}}{\pi} \int_0^{x_0} \frac{(\mathrm{d}/\mathrm{d}t)(Ue\delta + f)}{(x-t)\mathrm{E}\sqrt{x_0 - t}} \, \mathrm{d}t \tag{22}$$

This is the interacting boundary-layer equivalent of the pressure equation, Eq. (18), used in the triple-deck model and replaces Eq. (14) in the Veldman-Davis method. Note that x_0 has no direct relationship to the separation point in the interacting boundary-layer model; it is merely the location of the initial square root pressure gradient singularity in the outer inviscid flow. Rothmayer and Davis[10] showed that when x_0 was placed in a true constant-pressure region then massive (i.e., open-eddy) separation solutions could be found. Those solutions were regular at the point x_0 and were independent of the location of x_0, as long as x_0 was placed in a true constant-pressure region. It must be stressed that Eq. (22) implies that a Kirchhoff free-streamline model is being used for the eddy downstream of x_0. The method of Rothmayer and Davis[10] merely addresses the viscous solution controlling the flow near the separation point; it does not say anything about the consistency or correctness of the downstream eddy model. Their model provided very efficient calculations of laminar separation. It is believed that this occurred because Eq. (22) has an initial pressure gradient singularity of the form required to initiate the triple-deck free interaction. In the method of Rothmayer and Davis,[10] this free interaction is established quite rapidly, usually within the first 10 iterations, and it then serves to control the overall inviscid eddy.

Two obvious criticisms may be leveled on the method of Rothmayer and Davis.[10] The first is that many separation bubbles occur over bluff bodies, whereas their method treats only the thin airfoil case. The second objection is that it imposes an arbitrary Kirchhoff free-streamline model for the eddy. The first objection has been addressed by Rothmayer.[11] By using a conformal mapping procedure, Rothmayer[11] was able to construct bluff-body analogs of both the Cauchy-Hilbert model [Eq. (11)] and the Rothmayer-Davis[10] model [Eq. (22)]. Rothmayer's[11] bluff-body interacting boundary-layer theory was derived for both the symmetric and asymmetric cases (applicable to lifting airfoils) and yielded a formulation that preserved the Veldman-Davis algorithm. The present study seeks to address the second objection—correcting the Kirchhoff wake model. In the Rothmayer-Davis[10] method, the wake model requires a constant eddy pressure. However, the eddy pressure may be set to any desired value by simply changing the freestream dynamic pressure perceived by the body-scale flow. This effect would be due to a modification of the flow near the body by the large downstream eddy (see Smith[8]). Changing the eddy pressure completely changes the interacting boundary-layer solution. When the eddy pressure is set correctly, the Rothmayer-Davis[10] method is found to give very good results. For example, moderately good agreement has been found when comparisons were made with Navier-Stokes calculations at Reynolds numbers of 600 for flow past a circular cylinder (see the comparison of Rothmayer[11] results with the full Navier-Stokes calculations of Fornberg[19] in Ref. 11). At higher Reynolds numbers, the interacting boundary-layer method is found to be in excellent agreement with experimental observations of flow past a circular cylinder (see Barnett[20]). However, in all of these studies, the eddy pressure is an arbitrary parameter that must be set from a priori computational or experimental knowledge of the eddy properties.

V. New Interacting Boundary-Layer Method

The present study seeks to address the calculation of laminar separation bubbles in the wake inflation/deflation regime when the eddy does not have a constant pressure. Here, the length of the separation bubbles are on the scale of the airfoil chord length and may vary from extremely small trailing-edge separation bubbles to bubbles many airfoil chords in length. Once the separation bubble leaves the trailing-edge triple-deck scale, the strong viscous-inviscid interaction becomes confined to the vicinity of the separation point (and possibly the reattachment point). The flow near separation is a free viscous-inviscid interaction that is largely independent of the rest of the solution. The downstream eddy is fixed by the separation free interaction via the mechanism of Cheng and Smith.[7] As in the study of Rothmayer and Davis,[10] it is important to initiate and to encourage the free interaction, since it controls much of the overall separation bubble solution. At the same time, it is desired to calculate the entire separation bubble, up to and past the reattachment point. This is accomplished by modifying the original method of Rothmayer and Davis[10] to include the correct eddy pressure variations, as opposed to artificially imposing a constant eddy pressure.

The method used here is illustrated in Fig. 3. A mixed boundary value formulation is again used for the outer inviscid flow, as follows:

$$(u,v) \sim (1,0) + Re^{-1/2}(u',v') + \dots \tag{23}$$

with

$$v'(x,0) = \frac{\mathrm{d}}{\mathrm{d}x}(Ue\delta + f), \qquad x < x_0 \tag{24}$$

$$u'(x,0) = Re^{1/2}(Ue - 1), \qquad x > x_0 \tag{25}$$

Equation (25) replaces the constant eddy pressure assumption of Eq. (20) with the correct eddy pressure variation. This eddy pressure now must be calculated from a numerical solution within the eddy. The mixed boundary value problem is solved using the generalized method of Cheng and Rott.[15] A complex perturbation velocity is defined as

$$V \sim 1 + Re^{-1/2}V'(z) + \dots \tag{26}$$

where $V(z)$ is the complex velocity and

$$V' = u' - iv' \tag{27}$$

The boundary conditions on V' switch from specifying its real part u' to specifying its imaginary part v' along the x axis. This problem is converted to a direct problem by defining the following function:

$$F(z) = \frac{V'(z)}{H_0(z)} \tag{28}$$

where

$$H_0(z) = i(z - x_0)^{1/2} \tag{29}$$

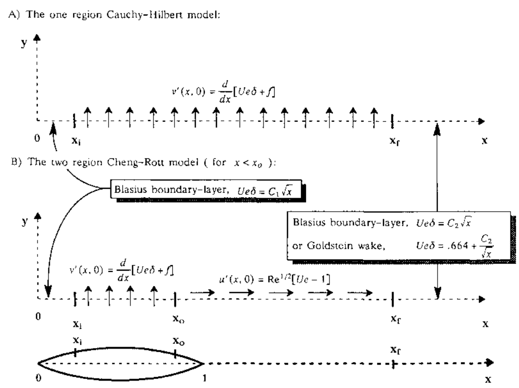

Fig. 3 Boundary conditions for the interacting boundary-layer models.

Because $H_0(z)$ switches from pure real to pure imaginary in crossing x_0, the boundary conditions on $F(z)$ become

$$q(x) = \text{Im}[F(z)]_{y=0} = \frac{v'(x,0)}{\sqrt{x_0 - x}}, \qquad x < x_0 \tag{30}$$

$$q(x) = \text{Im}[F(z)]_{y=0} = \frac{-u'(x,0)}{\sqrt{x - x_0}}, \qquad x > x_0 \tag{31}$$

Note that, whereas the boundary conditions for V' are on both the real and imaginary parts of $V'(z)$, the boundary conditions for $F(z)$ are on only the imaginary part of $F(z)$. This is simply a direct problem for $F(z)$. At this point, an analytic function must be found for $F(z)$ that satisfies Eqs. (30) and (31). Analyticity for $F(z)$ implies analyticity for $V'(z)$, which in turn satisfies the requirement that u' and v' are governed by Laplace equations. The solution for $F(z)$ is simply the classical Cauchy-Hilbert solution,

$$F(z) = \frac{1}{\pi}\int_0^\infty \frac{-q(t)}{z - t}\,\mathrm{d}t \tag{32}$$

Substituting for $F(z)$ and $q(x)$ and rearranging the above equation gives

$$Ue = 1 + \frac{\sqrt{x_0 - x}}{\pi}\left\{Re^{-1/2}\int_0^{x_0}\frac{(\mathrm{d}/\mathrm{d}t)(Ue\delta + f)}{(x-t)\sqrt{x_0 - t}}\,\mathrm{d}t + \int_{x_0}^{\infty}\frac{1 - Ue(t)}{(x-t)\sqrt{t - x_0}}\,\mathrm{d}t\right\} \qquad \text{for} \quad x < x_0 \tag{33}$$

In addition, an equation may be generated for δ when $x > x_0$ (see Ref. 10). Even though this new equation for δ may be used for calculating the flow past x_0, a simpler model will be used in this study. In the present study, Eq. (33) is applied for $x < x_0$ and Eq. (11) will be used for $x > x_0$. This model will be referred to as a two-region Cheng-Rott model. The model using Eq. (11) everywhere will be called the Cauchy-Hilbert model.

In implementing either Eqs. (11) or (33) in the Veldman-Davis method, the boundary-layer integration is started from an initial point x_i (see Fig. 3). The flow upstream of x_i is assumed to take the Blasius solution form, with the constant C_1 determined from matching with the first solution point. This may be further corrected to reflect the true Falkner-Skan form.[11] Such a correction appears to be unnecessary for the present problem, so long as C_1 is allowed to vary. The equations at x_i are assumed to be self-similar but with an unknown pressure gradient. The solution at the first point then is coupled with the assumed analytic solution upstream of x_i. In the numerical solution, x_i is placed quite close to the leading edge ($x_i \sim 0.01$) and so the above assumptions are expected to be accurate. Details of this procedure and the rest of the Veldman-Davis algorithm may be found in Ref. 13. The boundary layer is calculated numerically from x_i to x_f. Past x_f, a Blasius boundary-layer form is assumed for the splitter plate calculations and a Goldstein wake form is assumed when the splitter plate is removed. Again, the unknown constant C_2 is directly coupled into the solution at the last grid point, although self-similarity is not assumed. The *only* difference between the two-region Cheng-Rott model and the Cauchy-Hilbert model is that Eq. (33) is used in place of Eq. (11) for $x < x_0$. Otherwise, the two methods are identical. It should be noted that, for separation bubbles larger than about one chord length, the FLARE[21] approximation was used within the eddy to stabilize the method. The FLARE approximation was not needed for separation bubbles shorter than one chord length. The use of the FLARE approximation will be discussed further in the next section.

VI. Biconvex Airfoil with a Splitter Plate

The geometry of the biconvex airfoil is shown in Figs. 1 and 3. The airfoil extends from 0 to 1 along the x axis and is given

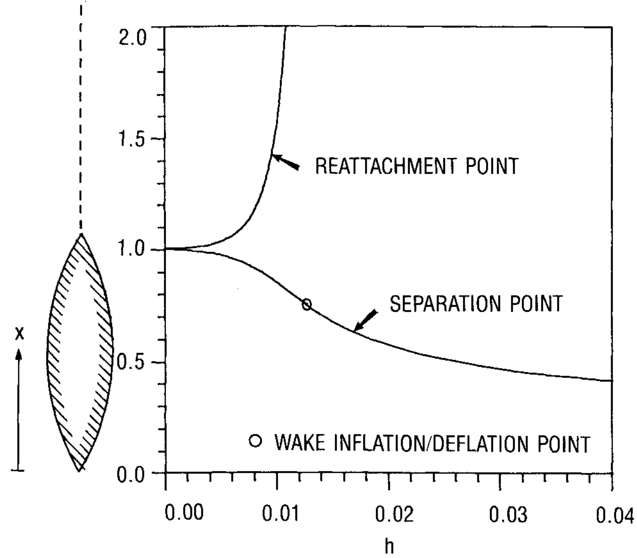

Fig. 4 Triple-deck separation and reattachment points for a biconvex airfoil.

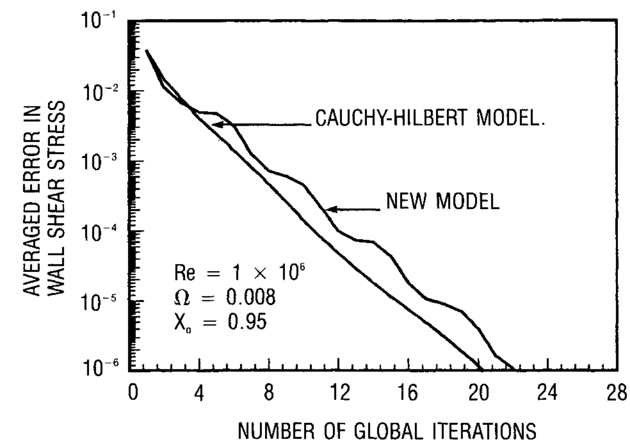

Fig. 5 Convergence rate comparison for an attached flow.

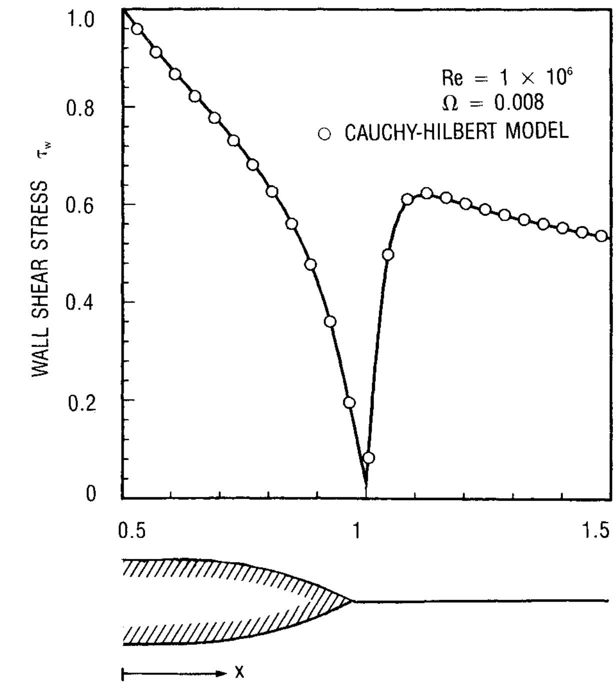

Fig. 6 Wall shear stress for an attached flow.

by

$$f(x) = Re^{1/2} 4\Omega x(1 - x) \tag{34}$$

where Ω is half the actual thickness-to-chord ratio. A splitter plate is placed from the trailing edge, $x = 1$, to infinity along the x axis. On the splitter plate $f = 0$, Eq. (6) is used for the wall boundary condition, and a Blasius form is assumed for the downstream analytic solution past x_f (see Fig. 3). Figure 4 shows the triple-deck predictions of the separation bubble growth for this airfoil, when the theory of Cheng and Smith[7] is used (see Rothmayer and Davis[22] for the equivalent interacting boundary-layer results). In Fig. 4, h is an airfoil thickness that has been scaled to $\mathcal{O}(Re^{-1/16})$ and is not directly related to Ω. The separation bubble starts out as a small trailing-edge bubble and rapidly grows to a large-scale separation bubble as the inflation point is approached. The following interacting boundary-layer calculations concentrate on the body-scale separation bubbles and on the approach to the inflation point.

Figures 5–7 show a comparison of the Cauchy-Hilbert method with the two-region Cheng-Rott model for this airfoil when $\Omega = 0.008$ or a 1.6% total thickness-to-chord ratio. As may be seen in Fig. 6, the flow is attached but is quite close to being separated near the trailing edge. Figure 5 shows that both methods converge at about the same rate and do so quite rapidly. The wall shear stress and pressure coefficient of the two methods are compared in Figs. 6 and 7, respectively. In both cases, the results for the Cauchy-Hilbert method and the two-region Cheng-Rott method are found to be in complete agreement. In the two-region Cheng-Rott method, x_0 was placed at 0.95. In Figs. 6 and 7, the solution is seen to be regular at x_0.

For the attached flow discussed above, the Cauchy-Hilbert method and the two-region Cheng-Rott method converged at about the same rate and yielded the same results. The picture is quite different for the two-chord-length separation bubble in Figs. 8–10. For this case, the airfoil thickness is $\Omega = 0.028$, a 5.6% total thickness-to-chord ratio, and from Fig. 8 the two-region Cheng-Rott method is seen to converge roughly two to three times faster than the Cauchy-Hilbert method. The convergence rate ratio appears to be asymptoting to about two as the number of iterations becomes large. This trend continues for the larger separation bubbles to be discussed later.

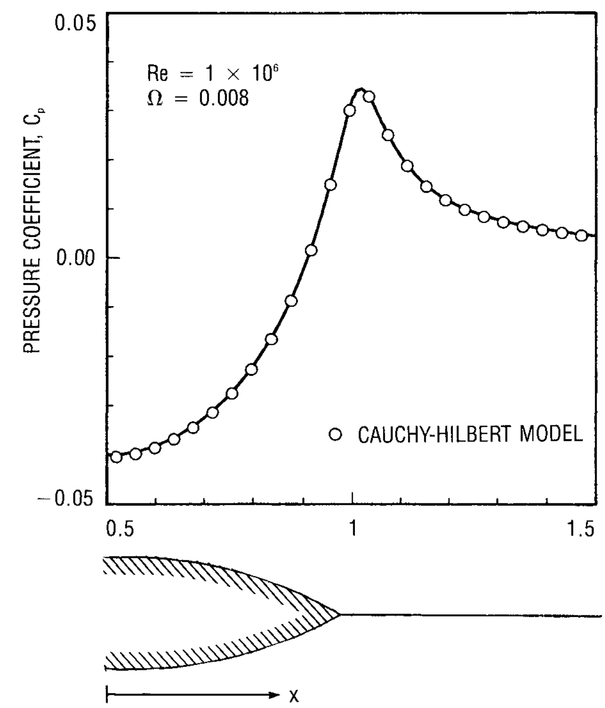

Fig. 7 Pressure coefficient for an attached flow.

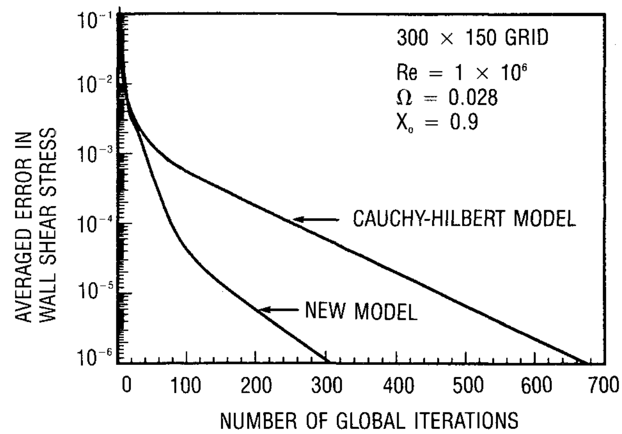

Fig. 8 Convergence rate comparison for $\Omega = 0.028$.

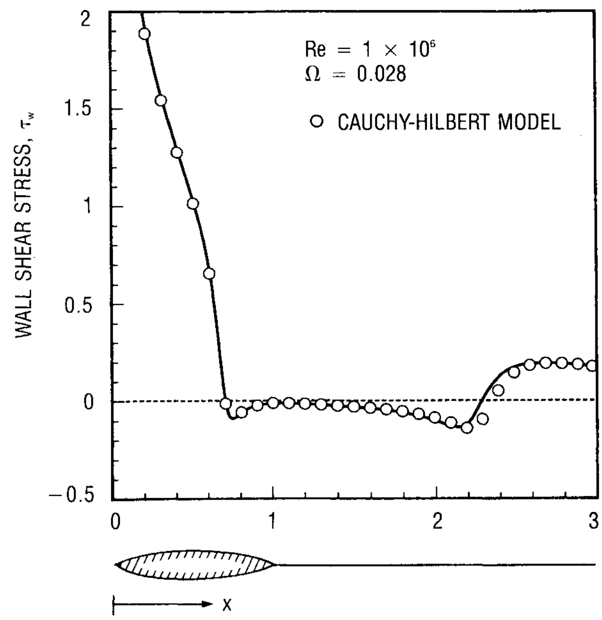

Fig. 9 Wall shear stress for $\Omega = 0.028$.

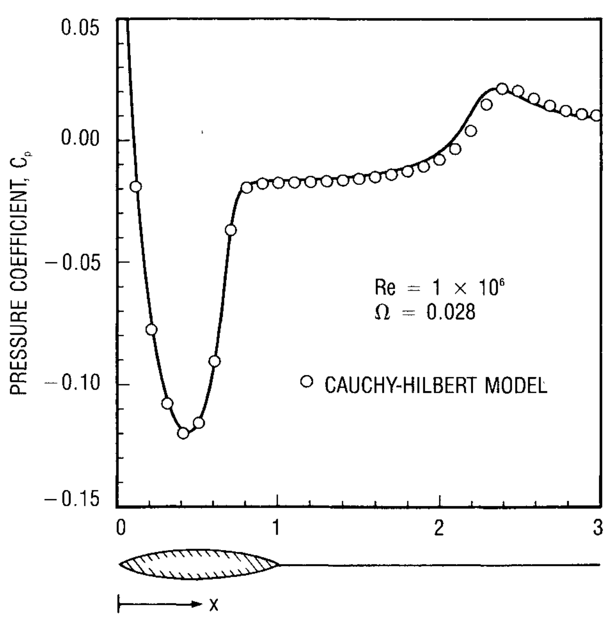

Fig. 10 Pressure coefficient for $\Omega = 0.028$.

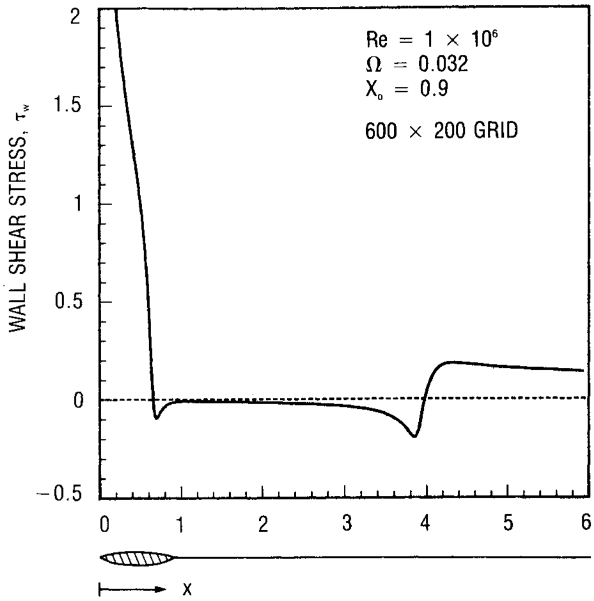

Fig. 11 Wall shear stress for Ω = 0.032.

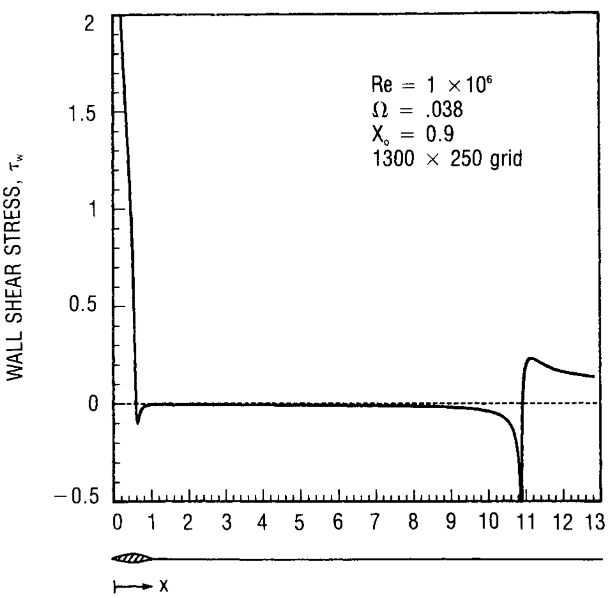

Fig. 12 Wall shear stress for Ω = 0.038 (partially converged, see Sec. VI).

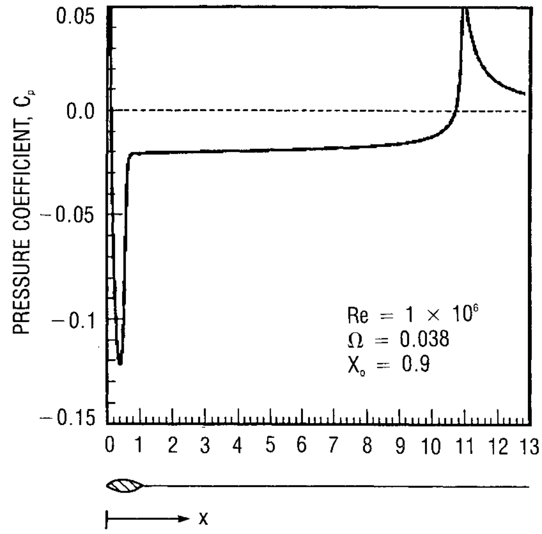

Fig. 13 Pressure coefficient for Ω = 0.038 (partially converged, see Sec. VI).

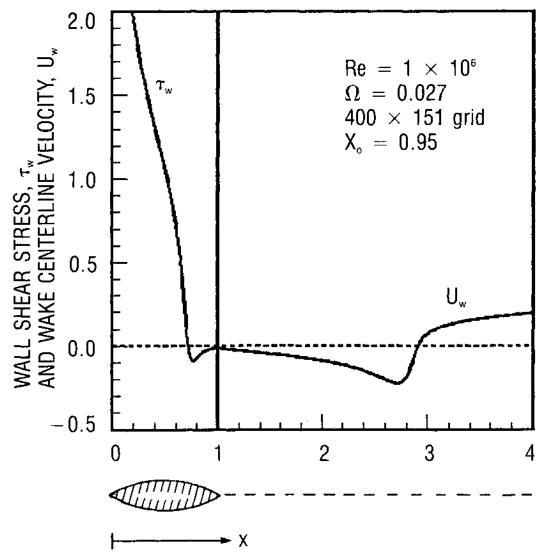

Fig. 14 Wall shear stress and wake centerline velocity for a biconvex airfoil with a symmetric wake.

As the size of the separation bubble increases, the convergence rate for both the Cauchy-Hilbert and the two-region Cheng-Rott methods tend to decrease rapidly. However, the two-region Cheng-Rott method always converges slightly better than twice as fast as the Cauchy-Hilbert method. Again, from Figs. 9 and 10, the wall shear stress and pressure coefficients of the two methods are in good agreement. Figure 10 shows that a constant pressure eddy is beginning to emerge as the size of the separation bubble is increased.

Figures 11 and 12 show wall shear stress calculations for two airfoils with larger separation bubbles, Ω = 0.032 (6.4%) and 0.038 (7.6%), respectively. Figure 11 is a completely converged solution, but Figs. 12 and 13 are not completely converged, although they are close to convergence. Figures 12 and 13 should be used to indicate trends and not be considered as final results. The partially converged case, Ω = 0.038, was run on a 1300 × 250 grid for about 2000 global iterations. The only part of the solution that remained unconverged was a narrow region near the reattachment point. The remaining error is due to a slow movement of the reattachment point downstream to its final location. The combination of the large grid and slow convergence times proved to be excessively time consuming for the minicomputer on which these calculations were run. The author believes that given a larger computer, the Ω = 0.038 case could be converged in a reasonable amount of time. The trends in Figs. 11 and 12 appear to qualitatively substantiate the wake inflation/deflation theory of Cheng and Smith.[7] As Ω is increased in these figures, the size of the eddy grows disproportionately to the increase in Ω. In addition, Fig. 13 shows that the eddy is maintaining a large constant-pressure region over virtually its entire length. However, when Fig. 13 is compared to Fig. 10, it is seen that the magnitude of the constant pressure within the eddy is asymptoting to a constant as the size of the separation bubble increases, which disagrees with the theory of Cheng and Smith.[7] There is one reservation that the author has about the present calculations—the use of the FLARE approximation in the separated region. For separation bubbles less than about one chord length, the FLARE approximation was not needed. However, for larger separation bubbles, an instability set in at the point of minimum wall shear stress and maximum displacement thickness. The FLARE approximation removed this instability. In all cases, for the biconvex airfoil, where the FLARE approximated results were compared with the results without the FLARE approximation, the two were found to be in complete agreement (i.e., for the smaller separation bubbles). However, the possibility that the FLARE approximation may be suppressing

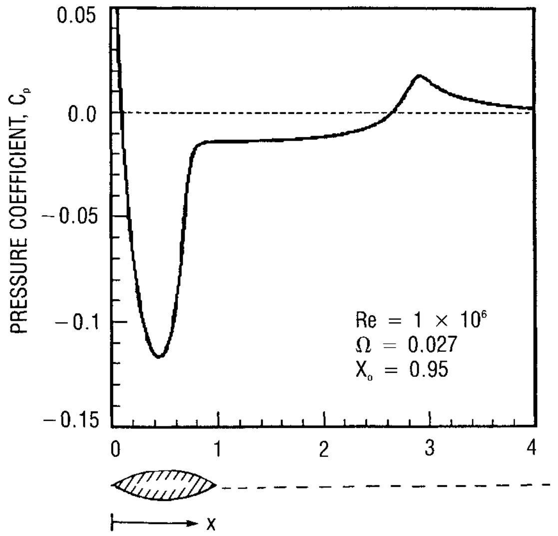

Fig. 15 Pressure coefficient for a biconvex airfoil with a symmetric wake.

the true physical solution for very large separation bubbles should not be ruled out, although the author feels that this is unlikely.

VII. Biconvex Airfoil with a Symmetric Wake

In this section, the splitter plate on the biconvex airfoil is removed. The no-slip boundary condition on the splitter plate [Eq. (6)] is replaced by the symmetry boundary condition [Eq. (7)]. In addition, the downstream analytic region is assumed to take the Goldstein wake form (see Fig. 3). The results for a biconvex airfoil with a two-and-one-half-chord-length separation bubble are shown in Figs. 14 and 15. It should be noted that the velocity levels within the eddy are quite low (~ 1-5% of the freestream level), and the eddy is again almost a constant-pressure, deadwater region. It is interesting that removing the splitter plate actually brings the wake pressure much closer to zero. Also, the splitter plate tends to suppress the wake inflation. This is shown by the fact that $\Omega = 0.028$ produced roughly a one-and-one-half-chord separation bubble when a splitter plate was used, but $\Omega = 0.027$ produced roughly a two-and-one-half-chord separation bubble when the splitter plate was removed.

VIII. Conclusion

In this study, a new interacting boundary-layer model has been developed to calculate incompressible attached and separated flows past thin airfoils aligned with an oncoming uniform stream. This new method was based on a consistent mixed boundary value formulation of the outer inviscid flow. No assumptions were made concerning the nature of the flow within the eddy. This new method was found to converge at the same rate as the classical Cauchy-Hilbert method for attached flows. However, for separated flows, the new method converged two to three times faster than the classical Cauchy-Hilbert method. The ratio of the convergence rate for the new method to the convergence rate for the Cauchy-Hilbert method was found to be largely independent of separation bubble size. Separated results calculated using this new method, as well as the Cauchy-Hilbert method, seem to support the overall conclusions of the wake inflation/deflation theory of Cheng and Smith,[7] although some discrepancies have been noted.

This study has addressed the relatively simple case of symmetric laminar separation past a symmetric airfoil. The extension to the asymmetric case and the bluff-body problem is believed to be straightforward, using the newly developed method of Rothmayer.[11] In addition, the present study may have considerable relevance to calculating turbulent separation past thin airfoils and bluff bodies. In turbulent separation, the separation bubbles are not constant pressure and are much smaller in length than the theoretically predicted large-scale laminar separation bubbles (see Barnett,[20] for examples). The present model would be ideally suited for calculating such body-scale separation bubbles with nonconstant-pressure eddies. In addition, it is believed by the author that further improvements in the convergence rates are possible. The use of different forms of the Cheng-Rott mixed boundary value formulation may serve to further accelerate the interacting boundary-layer calculations for separated flows.

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